MINIMAL DYNAMIC EQUILIBRIA

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Abstract. We define dynamic models as multiperiod models with no static representations and demonstrate that current prevalent asset pricing empirical implementations are inconsistent with dynamic equilibria. Specifically, empirical implementations are misspecified with respect to three essential asset pricing questions (TEQ): dependency on higher moments, complexity of risk premia, and mean-variance efficiency of the "market portfolio" (ability to proxy pricing kernels/SDFs). While we already know that "Merton" models, and their derivatives, differ from static models in all TEQ, we show that this is the case even the "minimal" dynamic equilibria.

Key Words: Minimal, Dynamic, Equilibrium, Higher Moments, Risk Premium, Pricing Kernel, SDF

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1   Introduction

Current asset pricing literature stands on two "legs," static and dynamic. Representatives of the first include Markowitz (1952), Sharpe (1963, 1964)-Lintner (1965)-Mossin (1966) single-period CAPM, and multifactor extensions [e.g., Fama and French (1992, 2015)], which, in fact, are single-period linear beta pricing models.[1] Representatives of the second include Samuelson (1969), Merton (1971, 1973), Lucas (1980), Cox, Ingersoll and Ross (1985a,b), Epstein and Zin (1989), Epstein (2001), and Hansen and Sargent (2001), which are multiperiod models with stochastic investment opportunities and with potentially, exchange, production, capital markets, intermediate consumption, incomplete information, ambiguity, and model uncertainty.[2] For brevity, we will call the latter multiperiod models and their derivatives "Merton models."

We call models static if they are either single-period or multiperiod with a single-period representation, that is, where in each and every period the analysis becomes a single-period one.[3] We call models "dynamic," if they are multiperiod with no single-period representation.

In the context of these two approaches to asset pricing, three essential questions have arisen (TEQ):

i.   Does the analysis map into a mean-variance (MV) one? Alternatively, is there dependency on higher moments?

ii.   Are risk premia (expected returns[4] in excess of the risk-free rate) "simple"? We call risk premia "simple" if they are similar to single-period ones, and "complex" if they are similar to those in "Merton models." The latter include additional term(s) relating to intertemporal rates of substitution.[5]

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[1] See also Merton (1972), Black (1972), Roll and Ross (1995), Kandel and Stambaugh (1995), Jagannathan and Wang (1996), Feldman and Reisman (2003), Bick (2004), Ukhov (2006), and Diacogiannis and Feldman (2013).
[2] See also Kreps and Porteus (1978), Dothan and Feldman (1986), Detemple (1986), David (1997), Feldman (2007), Björk, Davis and Landén (2010), Leisen (2016), and Leisen (2018).
[3] The latter are multiperiod models that map (degenerate) into single-period ones, or sequences of these, due to, for example, path independence, dependency on final outcomes only, Martingale representation methods, myopic preferences, or periodically independent returns. See Feldman (1992) findings on multiperiod equilibria with myopic optimal decisions.
[4] For brevity and simplicity, we will use the term "returns" also for "rates of return."
[5] These are optimal demands induced by stochastic changes in future investment opportunities, which Merton called "hedging demands."

iii.   Is the pricing kernel/stochastic discount factor (SDF)/market portfolio MV efficient?

The literature characterizes differences in answering these TEQ for static models and only a *subset* of "dynamic" models, the Merton models. Thus, the disparity/overlap in characterizations between static and other dynamic models, with respect to our TEQ, has not been fully explored. We know that the answers to TEQ for static models is *"yes," "yes,"* and *"yes,"* and for Merton models is *"no," "no,"* and *"no."* In this paper we ask, whether there exist dynamic models with answers to the TEQ that are different from the answers to Merton models, thus, more similar to the answers for static models.

Another way to describe the lacuna in the literature is as follows. We know that single-period representations of multiperiod models are sufficient for answering the TEQ with yes, yes, and yes, but we do not know if this is necessary.

Such characterizations are necessary, for example, for understanding the implications of empirical asset pricing implementations that, as a matter of common practice, use static models in multiperiod contexts. We note that these types of implementations are almost exclusively used in finance disciplines other than asset pricing. In other words, are there dynamic models that are consistent with the predominant asset pricing implementations?

Clearly, if there exist dynamic models with answers to the TEQ that are different from those of Merton models and similar to those of static models, they are likely to be among the "simplest" ones. Thus, we set, as our first objective, to identify a "minimal dynamic equilibrium" (MDE). Specifically, a dynamic model with the simplest structure in terms of number of periods, endowments, stochastic structure, information structure, and plausible preferences.

Our second objective is to answer our TEQ with respect to the MDE. In other words, can we identify MDE similar to static models in answering some or all three of the TEQ.

Our results are as follows. We identify an MDE that is minimal in all dimensions: it has MV risk-averse representative investor who maximize, over two periods, arithmetic mean returns (possibly of elliptical distribution functions) of investments in one riskless and numerous risky assets.

The answer to the first TEQ is ***no***: we find that in our MDE there is no riddance of the dependency

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on higher moments. That is, moments higher than variance do play a prominent role. The relevance of higher moments in risk premia was documented empirically [Harvey and Siddique (2000) and Dittmar (2002), for example]. Some single-period equilibrium models address this issue by defining preferences over higher moments [e.g., Kraus and Litzenberger (1976), Chabi-Yo (2012), and Chabi-Yo, Leisen and Renault (2014)]. Our findings demonstrate, however, that the role of the higher moments in forming equilibrium demands and prices in the face of stochastic investment opportunities is so natural that even within MDE, under MV preferences and elliptical return distributions, they conspicuously appear.

In this context, perhaps it is important to note the danger of misinterpreting the dependency on "only" instantaneous first two moments in continuous time formulations to be like the dependency on two moments in the static case. We must recognize the tradeoff between time and space in the continuous time case. The choice of different functions that instantaneous continuous time first two moments may assume allows inducing distributions with different specifications of higher moments, over any finite time interval.[6]

The answer to the second TEQ is **no**: we find that even MDE risk premia are not simple. They include a term, additional to the one in static models, which depends on the covariance between *prevailing* returns and *future* investment opportunities.

The answer to the third TEQ is **no**: we find that market portfolios are generally not MV efficient, thus, cannot serve as SDFs.

Furthermore, as, perhaps, an unexpected finding, we identify future market return's volatility as a priced factor and a component of the prevailing SDF (see also the following paragraph). This result has been confirmed empirically [see Chabi-Yo (2012)].

We offer insights into our results and what drives them. The first insight is that while, in general, dynamic equilibria offer a continuum of tradeoffs between income effects and substitution effects where

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[6] In Vasicek (1977), for example, over any finite time interval, a common factor is normally distributed; in contrast, in Cox, Ingersoll and Ross (1985b), prices/outputs could, conditionally, have a log normal distribution, and productivity factors a non-central chi-squared one.

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either or neither[7] effect dominates, under our MDE there is only a single such tradeoff within which it is the substitution effect that dominates.

The second insight is that Square Sharpe ratios sufficiently characterize future (stochastic) investment opportunities[8] with a "dimension" of square returns.[9] Moreover, covariances between returns and future investment opportunities shape risk premia with a dimension of cubic returns. This, in turn, induces a dependency on higher moments, elaborate risk premia, and MV inefficient market portfolios.

The third insight is our identification of an equilibrium relation between period 2 market expected returns and market return volatility, scaled by the market risk-aversion level, implying market volatility is priced. Moreover, this pricing is foreseen by, and has implications for, period 1 demands and prices. An increase (decrease) in the covariance between prevailing returns and future investment opportunities results in an increase (decrease) in prevailing expected returns as a pricing adjustment to added (reduced) risk.

We may characterize our MDE as a minimal extension of linear beta pricing models (such as the CAPM). However, we conjecture that our MDE choice could be considered as a *first* natural choice even over a larger set of commonly used asset pricing models (including overlapping generations, for example).

Section 2 identifies the MDE; Section 3 analyzes and characterizes the MDE; Section 4 offers a discussion; and Section 5 concludes.

2   MDE Identification

2.1   Selecting Preferences

MV preferences are an obvious candidate for preferences that are simple, plausible, and bring us closest to the static models. However, with MV preferences, as wealth increases, Arrow-Pratt's absolute risk aversion measure (ARA) increases as well. This is a property that describes none of us. Thus, MV

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[7] No dominating effect is only in the knife-edge case of logarithmic preferences that induce a single unit level of Arrow-Pratt's relative risk aversion (RRA). We rule out the logarithmic case from our MDE choices (please see below) because it induces equilibria that are iid repetitions of single-period equilibria. See, for example, Mossin (1968), Hakansson (1970), and Feldman (1992).

[8] Liu (2007) studies the case where investment opportunities are characterized by Sharpe ratios in a continuous time framework.

[9] For simplicity we use the term "dimension" to describe quadratic and cubic rates of returns, though rates of returns are unitless.

preferences "cannot" be over wealth.

We now examine potential model attributes and their implications for preferences. Path independence generally allows single-period representation. Single-period representation generally induces dependence on end of period wealth. Dependence on end of period wealth generally induces path independence—and so forth, creating a loop. One simple way of avoiding this "loop" is to allow returns' periodic dependence, which induces path dependency.

Moreover, geometric means, by construction, induce path independence, thus allowing for single-period representation. The choice of preferences over mean-returns, however, corresponds to an emphasis on intermediate periodic outcomes that may include or correspond to periodic consumption.

To summarize the implications of the above, we will not define preferences over wealth but over returns because of concerns about risk aversion. Further, due to concerns about path dependency, we will not define preferences over geometric mean returns. Therefore, we choose to define preferences over mean-returns; thus, our choice is MV preferences defined over mean-returns.[10]

2.2   The MDE

Pursuing utmost simplicity, consider a two-period, three-date, $t = 0,1,2$, Markowitz world with N risky securities, $N > 2$, and one riskless security. Risky securities are nonredundant, with finite moments. There exists a representative investor with MV preferences and Arrow-Pratt risk-aversion measure $\frac{A}{2}$. Security returns in excess of a constant risk-free rate are the $Nx1$ vectors R_t, prevailing through period t, $t = 1,2$. Security returns are driven by fundamentals/state variables/productivity factors. Period t, $t = 0,1$, first two, conditional on date t fundamentals, moments of returns are μ_t, Σ_t. Let $E_t(\cdot)$, $Var_t(\cdot)$, and $Cov_t(\cdot,\cdot)$ be the, conditional on date t fundamentals, expectations, variance, and covariance operators, respectively. The moments μ_t, $\mu_{t+1} = E_t(R_{t+1})$, and $\Sigma_t, \Sigma_{t+1} = \{Cov_t(R_{t+1,i}, R_{t+1,j})\}_{i,j=1,\dots,N}$, are $Nx1$ vectors and NxN matrices, respectively.

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[10] We argue below that our results are robust to the "less simple" case of compounded returns; see Section 4.3.

The representative investor period t portfolio's rate of return R_{Pt+1}, $t = 0,1$, is $R_{pt+1} = \theta_{t+1}^{\mathrm{T}} R_{t+1}$, where portfolio weights, θ_{t+1}, are $N x 1$ vectors. We note that portfolio weights are conditional on fundamentals.

Preferences are over portfolios' (arithmetic) mean excess returns over the two periods,[11] $\bar{R}_P \triangleq \frac{R_{P1}+R_{P2}}{2}$, where R_{Pt+1}, $t = 0,1$, is the representative investor's period t portfolio's rate of return. The representative investor trades off mean and variance by choosing portfolio weights $[\theta_1, \theta_2]$:

$$\underset{\theta_1,\theta_2}{\mathrm{Max}}\left\{\mathrm{E}_0(\bar{R}_P) - \frac{A}{2}\mathrm{Var}_0(\bar{R}_P)\right\}. \tag{1}$$

Note,

$$\mathrm{E}_t(R_{pt+1}) = \theta_{t+1}^{\mathrm{T}}\mu_{t+1},$$

$$\mathrm{Var}_t(R_{pt+1}) = \theta_{t+1}^{\mathrm{T}}\Sigma_{t+1}\theta_{t+1}, \tag{2}$$

where superscripts T denote the transpose operator. We note that all variables in the above equations are conditional on fundamentals.

Let θ_{Mt+1} $t = 0,1$, be the market capitalization weights, or the market portfolio weights. The market portfolio returns, are thus, R_{Mt+1}, $t = 0,1$, $R_{Mt+1} = \theta_{Mt+1}^{\mathrm{T}}R_{t+1}$.

We are now able to define the equilibrium.

Definition. Equilibrium: the representative investor holds optimal portfolio, which is the market portfolio.

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The existence of a representative investor implies that, by construction, the market clears.

3   MDE Characterizations

We are now ready to characterize the MDE.

3.1   Second Period Analysis

Proposition 1

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[11] As we are interested in the composition of risky assets holdings and risk premia, for simplicity, we define preferences over excess returns.

Conditional on period 1 fundamentals' realizations,

1. period 2 optimal portfolio weights, or market capitalization, are

$$\theta_2 = \frac{2}{A} \Sigma_2^{-1} \mu_2, \tag{3}$$

and

2. period 2 risk premia, i.e., expected returns in excess of the risk-free rate, are

$$\mu_2 = \frac{A}{2} \Sigma_2 \theta_2. \tag{4}$$

Proof. See Appendix.

Conditional on period 1 fundamentals' realizations, the period 2 problem becomes the classical single-period one.

We can now use Proposition 1's results to further characterize period 2 equilibrium, specifically the moments of the market portfolio return.

Corollary 1 to Proposition 1

Conditional on period 1 fundamentals' realizations,

1. period 2 market portfolio's expected return and variance are, respectively,

$$E_1(R_{M2}) = \frac{2}{A} \mu_2^T \Sigma_2^{-1} \mu_2, \tag{5}$$

$$\mathrm{Var}_1(R_{M2}) = \left(\frac{2}{A}\right)^2 \mu_2^T \Sigma_2^{-1} \mu_2, \tag{6}$$

and

2. period 2 market portfolio's square Sharpe ratio, S_2^2, is

$$S_2^2(R_1) = \mu_2^T \Sigma_2^{-1} \mu_2. \tag{7}$$

Proof. See Appendix.

We are now ready to highlight an equilibrium property of the MDE that relates period 2 market portfolio's Sharpe ratio and volatility. We will later demonstrate the implication of this property to the MDE equilibrium's dependence on higher moments.

If we substitute Equation (6) onto Equation (7), we get

$$S_2^2 = \left(\frac{A}{2}\right)^2 \text{Var}_1(R_{M2}), \tag{8}$$

demonstrating that, in equilibrium, the period two conditional square Sharpe ratio is equal to a unitless coefficient times the market portfolio return's variance, implying a dimension of square returns. Thus, we proved the following Corollary.

Corollary 2 to Proposition 1

The characterizations of period 2 square Sharpe ratios and (stochastic) investment opportunities have a dimension of "square returns." □

Thus, our MDE has the property that the unitless square Sharpe ratio of the market portfolio return, in equilibrium, becomes one to one with a variable representing square returns.

Another interesting insight conveyed directly by Equation (8) is the equilibrium relation between market Sharpe ratios and volatility. The higher the volatility, the higher the Sharpe ratio required to mitigate its effects on the derived utility. Moreover, this required mitigation is increasing in the representative investor's risk-aversion measure.

We now proceed to analyze the more interesting period, period 1. We start by defining, the covariance between period 1 returns and the future (period 2) investments opportunity set.

Definition. We define c_1, the covariance between period 1 returns and future (period 2) investment opportunity set, as

$$c_1 \triangleq \text{Cov}_0(R_1, S_2^2), \tag{9}$$

that is, $c_1 = \left\{\text{Cov}_0\left(R_{1,i}, S_2^2\right)\right\}_{i=1,\dots,N}$. □

We now characterize period 1, optimal portfolios, market capitalization, and risk premia.

3.2   First Period Analysis

Proposition 2

1. Period 1 optimal portfolio or market capitalization is

$$\theta_1 = \frac{2}{A}\Sigma_1^{-1}(\mu_1 - c_1). \tag{10}$$

2. Period 2 risk premia are

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$$\mu_1 = \frac{A}{2} \Sigma_1 \theta_1 + c_1. \tag{11}$$

Proof. See Appendix.

We now use Proposition 1's results to further characterize period 1 equilibrium, specifically, optimal portfolios' conditional Sharpe ratios and the stochastic conditional investment opportunity set.

Corollary 1 to Proposition 2

1. Conditional on period 1 realizations, the period 2 market portfolio's Sharpe ratio sufficiently characterizes the stochastic investment opportunity set of the MDE.

2. Higher moments than variance play a role in the MDE.

3. There is no degeneration of MDE risk premia to those of single-period models, and MDE risk premia depend on higher moments.

4. The MDE market portfolio is MV inefficient.

Proof. Equations (10) and (11) demonstrate that MDE's period 1 equilibrium demands and prices depend on the future (period 2) only through its Sharpe ratio (through their dependency on c_1).

This proves point 1 of the Corollary.

As c_1 is a covariance between returns and square Sharpe ratios—see the definition in Equation (9)—and as square Sharpe ratios in our MDE are proportional to the market portfolio variance, thus having a dimension of square returns—see Equation (8)—c_1 has a dimension of cubic returns, proportional to third moments. As c_1 is integral part of the MDE's demands and prices, see Equations (10) and (11), higher moments play a (substantial) role in the MDE.

This proves point 2 of the Corollary.

Equation (11) characterizes the MDE's risk premia as a sum of two addends. The first corresponds to single-period risk premia and is similar, for example, to those of the terminal period, period 2, where there are no future opportunities, see Equation (4). The second addend is $-c_1$, which has the dimension of third moments, as does the MDE's risk premia.

This proves point 3 of the Corollary.

Finally, Equation (10) demonstrates that the component c_1 takes optimal portfolio rules away from the single-period MV efficient demands. Only in the case where future investment opportunities are uncorrelated with prevailing ones will the period 1 market portfolio be MV efficient.

This proves point 4 of the Corollary.   *QED*

We note that c_1 corresponds, in Merton's terminology, to demands to hedge changes in future investments opportunities. We will now highlight how this corollary answers the TEQ. Point 2 established a "***no***" regarding the first TEQ (dependency on higher moments) overlap of MDE and static models.

Equation (11) shows that MDE's risk premia are also functions of c_1, which is an addend to the single-period risk premia. This demonstrates that MDE's risk premia do not degenerate to those of single-period ones. Moreover, we can also say that MDE's risk premia depend on higher moments because c_1 is a function of the third moment of returns. This establishes the second "***no***" regarding TEQ overlap.

The MDE's optimal demands or portfolio rules are equal those of the single-period ones only if $c_1 = 0$. Under non-zero c_1, the addend to single-period demands in the equation for optimal demands, Equation (10), can be viewed as the, so called, "hedging demands" term, that Merton coined to describe this part of optimal demands. These generally take the representative investor's portfolio away from the MV frontier, rendering their optimal portfolios MV inefficient. However, we may argue, following Merton, that in equilibrium it becomes optimal to "hedge" the changes in future investments opportunities. As these demands take the MDE (period 1) market portfolio away from the MV frontier rendering it "inefficient" and incapable of serving as the SDF, this establishes the third "***no***" regarding TEQ overlap.

Still, the MDE's optimal demands, as could be expected, are only a special case of demands in general dynamic equilibria. While an increase in risk aversion still reduces risky assets' holdings, it affects the single-period demands and "hedging demands" in equal proportion. Also, a higher positive (negative) covariance between prevailing returns and future investment opportunities would always reduce (increase) holding of risky assets.

Because in the MDE, R_{M1} and R_{M2} are the periodic market portfolio returns, we can now

specifically identify the SDF and demonstrate that the market portfolio is not the pricing kernel.

Rewriting Equation (11) using Equations (9), and (8), gives

$$\mu_1 = \frac{A}{2}\text{Cov}_0\left(R_1, R_{M1} + \frac{A}{2}\text{Var}_1(R_{M2})\right). \tag{12}$$

We note that the arguments of the covariance operator in Equation (12) are random variables, conditional on period 1 fundamentals' realizations.

Equation (12) identifies, up to a proportionality constant, the SDF as $R_{M1} + \frac{A}{2}\text{Var}_1(R_{M2})$.

We thus proved the following Corollary.

Corollary 2 to Proposition 2

The MDE SDF is, up to a proportionality constant, $R_{M1} + \frac{A}{2}\text{Var}_1(R_{M2})$. □

As the SDF includes an addend additional to the market portfolio's return, the market portfolio is not the pricing kernel.

4   Discussion

4.1   The Choice of MDE

In a different context, it might be interesting to study an MDE choice over preferences, assets, payoffs, strategies, constraints, time (discrete versus continuous), agents (short-lived, long-lived, overlapping generations), state space (finite versus infinite, discrete versus a continuum), markets (exchange, production, contingent claims), and markets (complete versus incomplete). In our pioneering study, however, we believe that starting with choices relevant to the most studied and implemented models is a good approach. After all, these models earned their place endogenously, in competition with other models. It is our conjecture that our MDE will maintain a primary position even after such studies are conducted. We expect, though, that studying additional MDE will bring new insights.

In identifying an MDE, we have two goals: minimal extension of static models and maximal simplification of dynamic models. There is a tradeoff, of course, between relevance and simplicity. Naturally, the MDE's choice is to opt for simplicity; relevance should be sought by other types of models.

An example of simplicity is our MDE state variable choice: the periodic returns. On the other hand, where we could enhance the structure with "no material cost," we chose to do so. Thus, we have multiple risky assets where three would be enough.

We note that adding periods to our MDE, or allowing stochastic equilibrium interest rates, will mitigate neither the three "no's" nor the market return's volatility's being a priced factor.

4.2   Defining MDE

A formal definition of MDE must involve numerous quantitative and qualitative dimensions. Further, it must be cardinal to facilitate aggregating over dimensions. We demonstrate below that ranking criteria are nonunique, subjective, and arbitrary. Thus, defining an MDE is illusive.

We demonstrate the illusiveness of defining MDE by examining a definition of one MDE aspect, preferences. Defining minimal preferences requires definitions of various preferences' dimensions.

- The distribution of coefficients of the Taylor series expansion of utility functions. Consider two distributions. One is $\{0.33+\Delta, 0.33, 0.33-\Delta\}$, the other $\{0.33, 0.33+2\Delta, 0.33-2\Delta\}$. Is a Δ advantage in the first and third coefficients enough to reconcile a 2Δ disadvantage in the second and third one? Ranking rules of such sequences are subjective, arbitrary, and nonunique.

- The number of cross-sectional and intertemporal risk-aversion coefficients. Merton models have one coefficient for any number of periods. Epstein-Zin preferences have two coefficients for any number of periods. Kreps-Porteus preferences have $2n-1$ coefficients for n periods. Ordinal ranking is natural here, but the required cardinal ranking is arbitrary and subjective.

- Whether preferences are time additive, of habit formation1, habit formation 2, etc. Even ordinal ranking, in this case, is nonunique, subjective and arbitrary.

- The stochastic nature of the utility and at the differential nature of the utility. Ranking would be nonunique, subjective, and arbitrary

Then, one has to create a weighing scheme over various dimensions of preferences, which is, again, nonunique, subjective, and arbitrary.

In addition to defining minimal preferences, one has to define other dimensions of the MDE and a weighing scheme across the various MDE dimensions. We trust that we have demonstrated that such a task is illusive as any outcome would be nonunique, subjective, and arbitrary.

4.3 Single-Period Representation of a Time Series

Considering a multiperiod problem, Cochrane (2014) aggregated a time series of payoffs into single points in the MV space, transforming the multiperiod model into a static one. While this might provide a useful, elegant transformation, the answers to the TEQ are clear (*yes*, *yes*, and *yes*) and it requires no further analysis.

4.4 Arithmetic Mean Returns versus Compounded Returns

An alternative specification to our MDE's criterion choice of arithmetic mean (excess) returns would have been compounded (excess) returns of the form, say,

$$(1 + R_{p1})(1 + R_{p2}) - 1 = R_{p1} + R_{p2} + R_{p1}R_{p2}. \tag{13}$$

We see that the outcome is similar to that of the MDE, with an added term of $R_{p1}R_{p2}$. The implication is that the phenomena we detected would persist and additional ones might arise due to the added term. An interesting empirical question is whether this added term has significant impact despite being an order of magnitude smaller.

5 Conclusion

While the object of finance models is a dynamic environment, prevalent asset pricing implementations are consistent with static models, with respect to three essential asset pricing questions (TEQ): dependency on higher moments, complexity of risk premia, and market portfolios being SDFs/pricing kernels/MV efficient. We already know that certain dynamic models, including Merton-type models and their various expansions, differ from static ones regarding the TEQ. In this paper, we aim to identify dynamic models that retain/capture the static properties regarding the TEQ. For this purpose, we

make the strongest assumptions that are likely to help capture the static properties and identify the "simplest/minimal" on all relevant dimensions, dynamic equilibria (MDE). We find that the MDE answer to the TEQ is **"no," "no,"** and **"no,"** respectively. Furthermore, the future volatility of MDE market portfolios' returns emerges as a pricing factor. This suggests that prevalent empirical asset pricing implementations are consistent only with static models.

Our MDE can be viewed as the minimal extension to most common asset pricing implementations – CAPM/linear beta pricing models. Asset pricing models, however, are not generally nested. It would, then, be interesting to keep searching for MDE. We conjecture, however, that it would be difficult to identify other MDE that capture the static essential asset pricing issues.

APPENDIX

Proof of Proposition 1

The period 2 problem, conditional on period 1 realizations, is

$$\underset{\theta_2}{\text{Max}}\left\{\text{E}_1\left(\frac{R_{p1} + R_{p2}}{2}\right) - \frac{A}{2}\text{Var}_1\left(\frac{R_{p1} + R_{p2}}{2}\right)\right\}. \tag{A1}$$

Conditional on period 1 fundamentals' realizations, we denote period 2 utility and period 2 derived utility (or indirect utility function) as $U_1(\theta_2)$ and J_1 respectively, we can define

$$U_1(\theta_2) \triangleq \text{E}_1\left(\frac{R_{p1}+R_{p2}}{2}\right) - \frac{A}{2}\text{Var}_1\left(\frac{R_{p1}+R_{p2}}{2}\right), \tag{A2}$$

and conditional on period 1 fundamentals' realizations

$$J_1 \triangleq \underset{\theta_2}{\text{Max}}\left\{\text{E}_1\left(\frac{R_{p1} + R_{p2}}{2}\right) - \frac{A}{2}\text{Var}_1\left(\frac{R_{p1} + R_{p2}}{2}\right)\right\} \tag{A3}$$

or

$$J_1 \triangleq \underset{\theta_2}{\text{Max}}\{U_1(\theta_2)\}. \tag{A4}$$

We can rewrite Equation (A1) as

$$J_1 = \underset{\theta_2}{\text{Max}}\left\{\frac{1}{2}R_{p1} + \text{E}_1\left(\frac{1}{2}R_{p2}\right) - \frac{A}{2}\text{Var}_1\left(\frac{1}{2}R_{p2}\right)\right\}, \tag{A5}$$

which, for finding optimal portfolio weights, is equivalent to

$$\underset{\theta_2}{\text{Max}}\left\{E_1\left(\frac{1}{2}R_{p2}\right) - \frac{A}{2}\text{Var}_1\left(\frac{1}{2}R_{p2}\right)\right\}. \tag{A6}$$

Thus, conditional on period 1 realizations, the period 2 problem becomes a standard single-period MV problem, and optimal portfolio weights are the argmax of the solution to the following problem:

$$\underset{\theta_2}{\text{Max}}\left\{E_1\left(\frac{1}{2}\theta_2^T R_2\right) - \frac{A}{2}\text{Var}_1\left(\frac{1}{2}\theta_2^T R_2\right)\right\}. \tag{A7}$$

The first-order condition is

$$\frac{\partial}{\partial\theta_2}\left[E_1\left(\frac{1}{2}\theta_2^T R_2\right) - \frac{A}{2}\text{Var}_1\left(\frac{1}{2}\theta_2^T R_2\right)\right] = 0 \tag{A8}$$

or

$$\frac{1}{2}\mu_2 - \frac{A}{4}\Sigma_2\theta_2 = 0 \tag{A9}$$

or

$$\theta_2 = \frac{2}{A}\Sigma_2^{-1}\mu_2. \tag{A10}$$

Because the second-order conditions are satisfied, θ_2, defined in Equation (A10), are period 2 optimal portfolio, or market portfolio, or market capitalization, weights vector.

This proves point 1 of Proposition 1.

Rearranging Equation (A10) yields

$$\mu_2 = \frac{A}{2}\Sigma_2\theta_2, \tag{A11}$$

which are period 2 market risk premia.

This proves point 2 of Proposition 1. *QED*

Proof of Corollary 1 to Proposition 1

Period 2 market portfolio's conditional expected (excess) return is

$$E_1(R_{M2}) = \theta_2^T\mu_2 = \frac{2}{A}\mu_2^T\Sigma_2^{-1}\mu_2, \tag{A12}$$

where the second equality holds after a substitution using Equation (3).

Period 2 market portfolio's (excess) return variance is

$$\text{Var}_1(R_{M2}) = \theta_2^\mathsf{T} \Sigma_2 \theta_2 = \left(\frac{2}{A}\right)^2 \mu_2^\mathsf{T} \Sigma_2^{-1} \mu_2. \tag{A13}$$

Again, the second equality holds after a substitution using Equation (3).

This completes the proof of point 1 of the Corollary.

Now, use Equations (A12) and (A13) to obtain

$$S_2^2 = \frac{(\text{E}_1(R_{M2}))^2}{\text{Var}_1(R_{M2})} = \frac{(\mu_2^\mathsf{T} \Sigma_2^{-1} \mu_2)^2}{\mu_2^\mathsf{T} \Sigma_2^{-1} \mu_2} = \mu_2^\mathsf{T} \Sigma_2^{-1} \mu_2, \tag{A14}$$

which gives Equation (7).

This proves point 2 of the Corollary. *QED*

Proof of Proposition 2

Denoting J_1 as the (total) derived utility, or period 1 derived utility, the period 1 problem is

$$J_1 \triangleq \underset{\theta_1, \theta_2}{\text{Max}} \left\{ \text{E}_0 \left(\frac{R_{p1} + R_{p2}}{2} \right) - \frac{A}{2} \text{Var}_0 \left(\frac{R_{p1} + R_{p2}}{2} \right) \right\}, \tag{A15}$$

which is not directly amenable to be solved under the Bellman's principle of optimality.[12] Using the laws of total expectation and total variance, we rewrite the Equation (A15) problem. Basak and Chabakauri (2010) and Björk, Murgoci and Zhou (2014) presented solutions to the problem in a continuous time context. Malamud and Vilkov (2018) use Basak's insights to present a discrete time solution to a similar problem within an overlapping generations model.

$$\underset{\theta_1, \theta_2}{\text{Max}} \left\{ \text{E}_0 \left(\text{E}_1 \left(\frac{R_{p1} + R_{p2}}{2} \right) - \frac{A}{2} \text{Var}_1 \left(\frac{R_{p1} + R_{p2}}{2} \right) \right) - \frac{A}{2} \text{Var}_1 \left(\text{E}_1 \left(\frac{R_{p1} + R_{p2}}{2} \right) \right) \right\}. \tag{A16}$$

Using Equation (A2), we can rewrite the problem in Equation (A16) as

$$\underset{\theta_1, \theta_2}{\text{Max}} \left\{ \text{E}_0(U_1(\theta_2)) - \frac{A}{2} \text{Var}_0 \left(\text{E}_1 \left(\frac{R_{p1} + R_{p2}}{2} \right) \right) \right\}. \tag{A17}$$

Using the definitions in Equations (A4) and (A15), and period 2 optimal portfolio weights values, θ_2, which we already determined, see Equation (3), we can rewrite Equation (A17) as

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[12] Because expectation of a variance is not equal to variance of expectation, the Bellman equation loses its recursive property. See Basak and Chabakauri (2010).

$$\underset{\theta_1}{\text{Max}}\left\{E_0(J_1) - \frac{A}{2}\text{Var}_0\left(\frac{1}{2}\left(R_{p1} + E_1(R_{p2})\right)\right)\right\}. \tag{A18}$$

We proceed by calculating the value of each of the two addends of Equation (A18). We will, then, identify the optimal values for period 1 portfolio weights, θ_1.

We first identify the first addend of Equation (A18), J_1, using Equation (A5). We substitute into it period 2 optimal portfolio weights values as determined in Equation (A10) and, further, substitute Equations (7) and (8). We have

$$J_1 = \frac{1}{2}R_{p1} + \frac{1}{2}\frac{2}{A}S_2^2 - \frac{1}{2A}S_2^2 = \frac{1}{2}R_{p1} + \frac{1}{2A}S_2^2. \tag{A19}$$

Taking expectation of Equation (A19), we have

$$E_0(J_1) = \frac{1}{2}\theta_1^{\mathrm{T}}\mu_1 + \frac{1}{2A}E_0(S_2^2). \tag{A20}$$

Calculating the value of the second addend of Equation (A18) gives

$$\text{Var}_0\left(\frac{1}{2}\left(R_{p1} + E_1(R_{p2})\right)\right) = \text{Var}_0\left(\frac{1}{2}R_{p1} + \frac{1}{A}S_2^2\right)$$

$$= \frac{1}{4}\text{Var}_0(R_{p1}) + \text{Var}_0\left(\frac{1}{A}S_2^2\right) + \text{Cov}_0\left(R_{p1}, \frac{1}{A}S_2^2\right) \tag{A21}$$

$$= \frac{1}{4}\theta_1^{\mathrm{T}}\Sigma_1^{-1}\theta_1 + \text{Var}_0\left(\frac{1}{A}S_2^2\right) + \frac{1}{A}\theta_1^{\mathrm{T}}c_1.$$

The first equality holds because of substitutions following Equations (5) and (7), and the last equality holds because of the use of the definition in Equation (9).

Using the results in Equations (A20) and (A21), the first-order conditions become

$$\frac{\partial}{\partial\theta_1}\left[E_0(J_1) - \frac{A}{2}\text{Var}_0\left(\frac{1}{2}\left(R_{p1} + E_1(R_{p2})\right)\right)\right] = \frac{1}{2}\mu_1 - \frac{A}{2}\left(\frac{1}{2}\Sigma_1\theta_1 + \frac{1}{A}c_1\right)$$

$$= \frac{1}{2}\mu_1 - \frac{A}{4}\Sigma_1\theta_1 - \frac{1}{2}c_1 = 0. \tag{A22}$$

Rearranging Equation (A22), gives period 1 optimal portfolio, or market portfolio, or market capitalization, weights vector

$$\theta_1 = \frac{2}{A}\Sigma_1^{-1}(\mu_1 - c_1). \tag{A23}$$

This proves point 1 of Proposition 2.

Solving Equation (A23) for the market risk premia gives

$$\mu_1 = \frac{A}{2}\Sigma_1\theta_1 + c_1. \tag{A24}$$

This proves point 2 of Proposition 2. *QED*

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